PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A
MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Chaos in Geophysical Fluids I. General Introduction

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Chaos in geophysical fluids
I. General introduction†

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Irregular buoyancy-driven flows occur in the atmospheres and fluid interiors of the Earth and other planets, and of the Sun and other stars, where they influence and often control the transfer of heat. Their presence is manifest in or implied by a wide variety of observed phenomena, including external magnetic fields generated by self-exciting magnetohydrodynamic (MHD) dynamo action. Based on the laws of classical mechanics, thermodynamics and, in the case of electrically conducting fluids, electrodynamics, the governing mathematical equations are well known, but they are generally intractable owing to their essential nonlinearity. Computers play a key role in modern theoretical research in geophysical and astrophysical fluid dynamics, where ideas based on chaos theory are being applied in the analysis of models and the assessment of predictability. The aim of this paper is to provide a largely qualitative survey for non-specialists. The survey comprises two parts, namely a general introduction (Part I) followed by a discussion of two representative areas of research, both concerned with phenomena attributable to symmetry-breaking bifurcations caused by gyroscopic (Coriolis) forces (Part II), namely (*a*) large-scale waves and eddies in the atmospheres of the Earth, Jupiter and other planets (where, exceptionally, laboratory experiments have been influential), and (*b*) MHD dynamos. Various combinations of Faraday disc dynamos have been studied numerically as low-dimensional nonlinear electromechanical analogues of MHD dynamos, particularly in efforts to elucidate the complex time series of geomagnetic polarity reversals over geological time. The ability of the intensively studied Rikitake coupled disc dynamo system to behave chaotically appears to be a consequence of the neglect of mechanical friction, the inclusion of which renders the system structurally unstable.

1. Introduction

Flows in the atmospheres and interiors of the Earth and other planets and of the Sun and other stars are driven by buoyancy forces due to the action of gravity on spatial variations of density. These density variations are associated with temperature variations produced and maintained by differential heating and cooling, and they are modified by variations in pressure and chemical composition.

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Through their ability to transport ('advect' or 'convect') heat in both horizontal and vertical directions, such fluid flows influence and often control the overall heat balance and evolution of the systems within which they occur.

Buoyancy-driven flows derive their kinetic energy from gravitational potential energy, with heavier fluid tending to sink and lighter fluid to rise. Concomitant advection of heat in the vertical is always in the upward direction (unless the thermal coefficient of cubical expansion of the fluid is negative). Upward advection of heat by large-scale waves and jet streams in the lower reaches of the Earth's atmosphere – the troposphere – helps to maintain the vertical temperature gradient there at a value somewhat less than the adiabatic value. This in turn influences the scales and other characteristics of tropospheric flows, thereby providing a dynamically important feedback mechanism. Horizontal advective heat transfer by this flow – and also by the currents driven in the underlying oceans by atmospheric surface winds and by buoyancy forces – keeps the equator-to-pole temperature difference at the Earth's surface at a fraction of the radiative equilibrium value that would otherwise obtain in order to balance differential solar heating.

The two general conditions for the occurrence of stable hydrostatic equilibrium – with no fluid flow and buoyancy forces everywhere balanced by pressure gradients alone (see (2.1) below) – are quite strict. The first is that density gradients nowhere possess a horizontal component, for otherwise fluid elements would experience gravitational torques, which cannot be balanced by pressure gradients. The second is that the vertical density gradient is either (*a*) 'bottom heavy' (i.e. the 'potential density' nowhere increases upwards (implying that the actual density everywhere decreases upwards at a rate less than the so-called 'adiabatic gradient', which vanishes when the fluid is incompressible), or (*b*) 'top-heavy' but of insufficient magnitude for buoyancy forces to be able to promote instability against inhibiting effects due to viscosity, thermal conduction and radiation, and also to gyroscopic (Coriolis) forces when the whole system is in general rotation, and to Lorentz forces when the fluid can conduct electricity and magnetic fields are present. When the first of these conditions is satisfied but the second is not, convection of the Rayleigh–Bénard type occurs, such as the flow that gives rise to the 'granular' appearance of the solar photosphere. Another example of Rayleigh–Bénard convection is the very slow flow – centimetres per year – in the highly viscous mantle of the Earth that geophysicists invoke to account for heat transfer there and also for various tectonic and other processes manifest in geophysical and geological data. When the second condition is satisfied but first is not, as in extensive regions of the atmospheres of the Earth and other planets, where the vertical density gradient is 'bottom heavy' but there are impressed horizontal density gradients, flow must always occur no matter how small the magnitude of these gradients. Many factors determine the form and speed of the flow, and in the important case when Coriolis forces exert a dominant influence, the process of 'sloping convection' can occur, in which the typical trajectory of a fluid element is inclined at an angle to the horizontal less than the slope of the surfaces of equal potential density. Through their differential action on the axial and non-axial components of the flow velocity vector, Coriolis forces tend to promote sloping convection, which is the dominant large-scale dynamical process in the extra-tropical regions of the Earth's atmosphere.

Irregularity characterizes the spatio-temporal behaviour of geophysical (and

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

astrophysical) fluids, as evinced by both direct and, more commonly, indirect observations (see below). Attempts to interpret the observations in terms of underlying fluid-dynamical processes or to predict future behaviour require in the first instance the separation of manifestations in the data of effects due to random forcing (noise) of various kinds from any that might be attributable to deterministic chaos. The 'predictability horizon' of a system exhibiting irregular behaviour depends, among other things, upon the extent to which the underlying processes are dominated by deterministic chaos rather than by noise. In principle it should be possible to identify deterministic chaos in observational time series, by using sophisticated modern methods (see Drazin & King 1992; Ghil *et al.* 1991; Moon 1987; Mullin 1993; Ott 1993; Smith 1988, 1992, this volume; Thompson & Stewart 1986) developed for adducing evidence of low-dimensional attractors in the data. This can often be done in investigations of well-controlled laboratory systems (see Guckenheimer & Buzyna 1983; Haken 1981; Mullin 1993; Read 1993; Read *et al.* 1992), but it is more difficult or even impossible in the study of observational time series from geophysical and astrophysical fluids and other natural systems, owing to the inadequate length of available time series and to measurement errors. So other kinds of evidence for chaos must be sought, such as that afforded by constructing models of the prototype with varying degrees of simplification (see § 2 below) and analysing their behaviour. The clear identification in the prototype of régimes of behaviour determined by bifurcations at critical values of key parameters as revealed by the models would be evidence for chaos.

Optimism – albeit limited – shown by workers concerned with basic problems of predictability of the terrestrial atmosphere (see Ghil *et al.* 1991; Lorenz 1963, 1967, 1980, 1993; Mason *et al.* 1986; Monin 1972; Mullin 1993; Nicolis & Nicolis 1987; Palmer *et al.*, this volume; Thompson 1957, 1988; Webster & Keller 1975; White 1990) derives to some extent from the practical experience of weather forecasters and climatologists. Also influential according to the literature have been laboratory experiments such as those carried out on sloping convection in a rotating fluid annulus subject to a steady axisymmetric temperature gradient (Hide 1953). In these experiments key parameters were identified and several flow régimes of varying degrees of spatial and temporal complexity were first delineated, ranging from steady axisymmetric flow through periodic and quasi-periodic non-axisymmetric flows to highly irregular 'geostrophic turbulence'. The quasi-periodic wave-like flows discovered in these experiments and termed 'vacillation' have attracted some attention, but they have not yet been reproduced in satisfactory detail in numerical models (see Hignett *et al.* 1985; Lorenz 1963*b*; Quinet 1973; Thompson 1988; White 1988). Elsewhere in this volume, Palmer *et al.* explain how modern ideas in the theory of deterministic chaos are guiding meteorologists armed with very powerful computers and highly sophisticated numerical models of the troposphere and the overlying stratosphere in their efforts to monitor global-scale atmospheric flow and forecast its future behaviour. Underlying such exercises is the implicit recognition of the possibility 'intransitivity' and 'multiple equilibria' (in modern parlance) (see Hide 1953) as well as other manifestations of essentially nonlinear and possibly chaotic behaviour such as hysteresis (see Fultz *et al.* 1959) in atmospheric flows. (For further references to laboratory studies and related work, see Buzyna *et al.* (1984), Ghil & Childress (1987), Hart (1986), Hide (1977), Hide & Mason (1975), Lewis (1992), Read (1988), Read *et al.* (1992), Smith (this volume) and articles in Corby (1969), Hopfinger (1992), King & Mobbs (1991), Mullin (1993) and Roberts & Soward (1978).)

Owing to public demands for weather and climate forecasts, meteorologists enjoy abundant observations of wind velocity, pressure, temperature and other variables of direct dynamical relevance, collected frequently and systematically at many levels over the whole globe, and made available rapidly in 'user friendly' form, making meteorologists the envy of workers in other areas of geophysical and astrophysical fluid dynamics. And for studying temporal fluctuations in one particularly important and revealing average property of the global atmospheric circulation, namely the total angular momentum, accurate surrogate data can be deduced from geodetic observations of fluctuations in the Earth's rotation vector (see Hide 1984; Hide & Dickey 1991; Rosen 1993). But in other areas of geophysical fluid dynamics such as the study of the atmospheres of other planets, observations are usually indirect in nature and of more restricted coverage in space and time, as they also are in studies of the oceans and of the Earth's atmosphere on the long timescales of interest to climatologists (see Berger *et al.* 1989; Ingersoll 1990; Ingersoll & Lyons 1993; James 1994; Neelin *et al.* 1994; Philander 1992; Saltzman & Verbitsky 1993; Sreenath 1993; Willebrand & Anderson 1993).

In the study of magnetohydrodynamic (MHD) flow in the Earth's liquid metallic core, where the main geomagnetic field is produced by self-exciting MHD dynamo action, the main observational data comes from magnetic measurements made at or near the surface of the Earth (see Jacobs 1987–91, 1994; Melchior 1986; also Part II). Magnetic observations also provide the basis for studies of MHD flows in the electrically conducting deep interiors of other planets such as Jupiter, Saturn, Uranus and Neptune and in the outer layers of the Sun (see Ness 1994; Proctor & Gilbert 1994; Sonett *et al.* 1991; Stevenson 1983). Additional data in the case of the Sun comprise a long time series of annual sunspot numbers, which exhibits nearly periodic oscillations with a dominant period of about 11 years. In the interpretation of these observations in terms of basic fluid-dynamical processes, highly simplified models analysed by methods suggested by chaos theory are being applied with some success (Platt *et al.* 1993; Ruzmaikin *et al.* 1994; Weiss 1990, 1993, this volume).

Many essentially nonlinear phenomena are encountered in astronomy and geophysics, including earthquakes, the prediction of which is the topic of another paper in this volume (M. Matsuzaki, this volume). In the present article, two topics in geophysical fluid dynamics are selected for more detailed discussion, namely the study of (*a*) large-scale motions in planetary atmospheres, where laboratory studies have played a useful role, and (*b*) planetary magnetism, with emphasis on polarity reversals of the main geomagnetic field where theoretical work has to rely on mathematical modelling alone. The article reflects my own personal scientific interests and point of view. In preparing it I have relied heavily on my research library, which includes many reprints of papers from a wide range of journals, which colleagues have kindly sent to me over a period of more than 40 years. A full bibliography citing all relevant original references would run to the editorially unacceptable length of more than 200 items, so I have included a selection of books, monographs and review articles where these original references can be found. Also it is impossible to reproduce here the many pictures and diagrams used to illustrate the lecture upon which this article is based.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

2. Basic equations and modelling

Applying Newton's laws of motion to a fluid element of unit volume which at time t is located at the general point whose vector position is $\boldsymbol{r}$ in the chosen inertial frame of reference gives the equation

$$\rho \mathrm{D}\boldsymbol{u}/\mathrm{D}t = -\nabla p - \rho \nabla V + \boldsymbol{F}. \tag{2.1}$$

Here ρ denotes the mass density of the fluid at P, $\boldsymbol{u}$ is the eulerian flow velocity, and the operator

$$\mathrm{D}/\mathrm{D}t \equiv \partial/\partial t + \boldsymbol{u} \cdot \nabla \tag{2.2}$$

is the 'substantial' time derivative following the moving fluid element. ∇p is the gradient of the pressure p, $-\nabla V$ is the acceleration due to gravity, and $\boldsymbol{F}$ represents all the other forces acting on the fluid element, including viscosity. In the case of electrically conducting fluids, with which the subject of MHD is concerned, $\boldsymbol{F}$ also includes the Lorentz force $\boldsymbol{j} \times \boldsymbol{B}$ where $\boldsymbol{j}$ is the electrical current density at P and $\boldsymbol{B}$ the magnetic field. Continuity of matter is expressed by the equation,

$$\mathrm{D}\rho/\mathrm{D}t + \rho \nabla \cdot \boldsymbol{u} = 0. \tag{2.3}$$

All these quantities, ρ, $\boldsymbol{u}$, p, V, $\boldsymbol{j}$ and $\boldsymbol{B}$ are in general functions of $\boldsymbol{r}$ and t. Equations (2.1) and (2.3) effectively comprise four scalar partial differential equations in twelve scalar unknowns, namely ρ, p and V and the individual components of the three-dimensional vectors $\boldsymbol{u}$, $\boldsymbol{B}$ and $\boldsymbol{j}$, so additional equations are needed. When ∇V is variable the law of gravity is used to relate ρ and V; and when ρ is variable an equation of state relating ρ at P to the pressure p, entropy S, composition K and temperature T, etc., is needed, together with transport equations for S, K, and T. These include the equation

$$(\partial/\partial t + \boldsymbol{u} \cdot \nabla)(\rho c T) = J \tag{2.4}$$

expressing the heat balance of a fluid element of unit volume at P if c is the specific heat. Here the term $\boldsymbol{u} \cdot \nabla(\rho c T)$ represents the contribution of advection to the heat balance and J represents all other contributions, including conduction and radiation.

Finally, in the case of MHD flows in electrically conducting fluids, such as those found in planetary interiors and in stars, we need the equations of electrodynamics applied to a moving medium. These relate $\boldsymbol{u}$, $\boldsymbol{j}$ and $\boldsymbol{B}$ implicit in (2.1) and bring in further variables such as the electric field vector $\boldsymbol{E}$ and the electric charge density θ at P. In the case of non-relativistic flows, $\boldsymbol{B}$ and $\boldsymbol{j}$ are related by Ampères law $\nabla \times (\boldsymbol{B}/\mu) = \boldsymbol{j}$ (where μ is the magnetic permeability) and the other equations of electrodynamics (expressing the laws of Gauss, Faraday, etc.), which lead to the following equation involving the time rate of change of $\boldsymbol{B}$:

$$\partial \boldsymbol{B}/\partial t - \nabla(\boldsymbol{u} \times \boldsymbol{B}) = -\nabla \times (\sigma^{-1} \nabla \times (\mu^{-1} \boldsymbol{B})) + \nabla \times \boldsymbol{Z} \tag{2.5}$$

(cf. Moffatt 1978; Parker 1979). Here σ and $\boldsymbol{Z}$ are defined by a generalized Ohm's law applied to a moving medium:

$$\boldsymbol{j} = \sigma[\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B} + \boldsymbol{Z}]. \tag{2.6}$$

The full set of nonlinear partial differential equation (PDEs) thus obtained is complete. In addition to c, μ and σ (see (2.3) and 2.5)), the equations include

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

other coefficients needed to specify the physical properties of the fluid at a general point P at time t such as the coefficient of viscosity, thermal coefficient of cubical expansion, ratio of principal specific heats, coefficients of thermal conductivity, chemical diffusion, dielectric constant, etc. The mechanical, thermal and electromagnetic boundary conditions under which these equations must be solved in specific cases are also well known, at least in principle. They usually amount to the requirement that all dependent variables and their derivatives as well as certain fluxes be continuous everywhere. It follows that theoretical fluid dynamics is not handicapped primarily by incomplete knowledge of the basic equations to be solved! It is the intractability of these equations owing to their essential nonlinearity that causes the main difficulties. Nonlinearity can arise in important cases from the boundary conditions or from spatial and temporal variability in the various parameters such as viscosity in the governing equations. More commonly however it is advection that it is responsible, as expressed by the terms $(\boldsymbol{u} \cdot \nabla)\boldsymbol{u}$, $(\boldsymbol{u} \cdot \nabla)\rho$, $(\boldsymbol{u} \cdot \nabla)(\rho c T)$ and $(\boldsymbol{u} \cdot \nabla)\boldsymbol{B}$ implicit or otherwise on the left-hand sides of (2.1), (2.3), (2.4) and (2.5) respectively. In spite of these difficulties, thanks to powerful modern computers and a wide range of new techniques in laboratory studies, where computers can be used in the control of apparatus and the analysis of data, the subject of fluid dynamics continues to develop as a lively branch of classical physics with many applications in engineering, geophysics and astronomy.

All mathematical modelling based on the above equations involves the simplification of both the equations and the boundary conditions governing the prototype being studied, and there is an art in constructing such models. In all cases the most sophisticated models such as those now used in numerical weather prediction involve minimal simplifications, still resulting in PDEs to be solved for all dynamical variables. At the other extreme are very highly simplified 'low-dimensional' models governed by ordinary differential equations (ODES) usually expressing the time evolution of mean quantities such as spatially averaged velocity, temperature, magnetic field, etc. (see Lorenz 1963*a*; Moore & Spiegel 1966). Such models might be trustworthy when they are derived by rational arguments from the full PDEs, but the most highly simplified models used in the study of the behaviour of nonlinear systems can be irrational ones, for they are often based to some extent on intuition, giving errors which are hard to assess. Analyses of such models provide results which might be helpful with diagnostic studies of more complicated systems, but they always carry a 'health warning'. Nevertheless, in the right hands findings based on low-dimensional models can be brought to bear on truly quantitative studies of the prototype, just as metaphysical ideas, though non-scientific, can influence truly scientific work! The most favourable situation arises when the investigator has at his disposal the results not only from numerical integrations of a range of mathematical models but also from relevant laboratory studies. Recent books (Cvitanović 1984; Haken 1981; Mullin 1993) discuss examples of thermally driven Rayleigh–Bénard convection and sloping convection (see § 1 above and Part II) and mechanically driven motions such as Taylor–Couette flow. Useful numerical integrations were not available before high-speed computers were introduced and guidance then had to be sought solely from laboratory experiments and from limited and sometimes misleading findings based on linearized models.

There is also an art in the exploitation of results from mathematical or laboratory models in research on much more complicated and uncontrollable natural systems. In his celebrated monograph on the general circulation of the Earth's atmosphere, Lorenz (1967) devotes a whole chapter to the discussion of the results of laboratory experiments on sloping convection in cylindrical apparatus (see also Lorenz 1993). To paraphrase his concluding remarks, the laboratory experiments tell us more about planetary atmospheres in general than about the Earth's atmosphere in particular. They indicate the variety of flow patterns that can occur and the conditions favourable to each of these. Perhaps the most important contribution of laboratory experiments to the theory of the general circulation of the terrestrial atmosphere has been the separation of essential considerations from the minor and irrelevant. Condensation of water vapour, for example, may yet play an essential role in the Tropics but in temperate latitudes it appears to be no more than a modifying influence, since systems occurring in the atmosphere, including even cyclones and fronts, are found in the laboratory system, where there is no analogue of the condensation process. Similar remarks apply to the topographic features of the Earth and the so-called 'beta' effect (which represents the latitudinal variation of the Coriolis parameter associated with the near-spherical shape of the Earth) now appears to play a lesser role than had once been supposed by theoreticians. Certainly a numerical weather forecast would fail if the beta-effect were disregarded, but the beta-effect does not seem to be required for the development of typical atmospheric systems. The experiments emphasize the necessity for truly quantitative considerations. At the very least these must be sufficient to place the Earth's atmosphere in one of the régimes discovered in the experiments.

Most of the experiments to which he refers were carried out in the early 1950s (for references see Corby (1969), Hide & Mason (1975)), but there have been many subsequent related studies (see Drazin & King 1992; Guckenheimer & Buzyna 1983; Hide *et al.* 1994; Hignett *et al.* 1985; Hopfinger 1991; King & Mobbs 1991; Lorenz 1963; Nicolis & Nicolis 1987; Ott 1993; Quinet 1974; Read 1993; Read *et al.* 1992; Smith 1988, 1992, this volume; White 1988, 1990). Modern work on sloping convection is based on the powerful combination of laboratory experiments and numerical models, and it is able to exploit some of the new methods of signal processing and time-series analysis to which chaos theory has given rise (see § 1 above). In such work, where in the search for 'low-dimensional attractors' in the output of transducers recording the behaviour of a system the acquisition of long time-series of high signal-to-noise ratio data is crucial, numerical models and laboratory models clearly have a key role to play. As already mentioned in § 1, observational time series from natural geophysical fluid systems are usually much too noisy and limited in duration for chaos theory to be directly applicable in their interpretation.

Just as in meteorology it is of interest to know what the atmospheric general circulation would be like in the absence of various complicating effects such as the presence of continents and oceans, water vapour in the atmosphere, time-varying thermal forcing, etc., in the boundary conditions, a central question in geomagnetism concerns the extent to which it might be necessary to invoke complex thermal, mechanical and electrical boundary conditions imposed on the Earth's liquid metallic fluid core by very slow flow occurring in the highly viscous overlying mantle in order to account for observations such as the highly variable frequency of polarity reversals of the main geomagnetic field. Should polarity

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

reversals be regarded as being 'forced' by irregular and fluctuating boundary conditions, or as an manifestation of 'free' instabilities that would occur under simple and fixed boundary conditions? Modern research indicates that details of the pattern of the geomagnetic field at the surface of the Earth are manifestations of both 'forced' and 'free' processes. Edmond Halley raised money from the Crown at the end of the seventeenth century for the purpose of mapping this pattern (see Chapman 1941) with a view to predicting future changes, before the invention of the marine chronometer effectively solved the main practical problem which Halley had in mind, namely the accurate determination of geographical longitude at sea.

The very existence of the durable Great Red Spot in the atmosphere of Jupiter – more than three centuries after its discovery by Robert Hooke in 1665 – and of other long-lived dynamical features in the atmospheres of the major planets must have implications for theories of atmospheric predictability. Why are there more of these markings in the Southern Hemisphere of Jupiter than in the Northern Hemisphere? What is the role of these eddies in the heat balance of the atmosphere? To what extent should the eddies be regarded as 'free' or 'forced', and what (if any) is their relationship to the field of transient eddies within which they are embedded? Can we infer the vertical structure of Jupiter's atmosphere (or even just its depth) from the flow at the observable upper level? These and many other related questions pose key problems in the essentially nonlinear dynamics of geophysical fluids.

3. Symmetry breaking, heat transfer by geostrophic and magnetostrophic flows, and elastoid oscillations

It is inconvenient to use (2.1) when dealing with fluid flow in planets and stars, which usually rotate rapidly relative to an inertial frame. Owing to the rotation of the solid Earth with a period of 24 hours, points on the equator move in space at a speed some 40 times the typical speed ($10\,\mathrm{m\,s^{-1}}$) of atmospheric winds relative to the Earth's surface. The corresponding factor for ocean currents is even higher, about 4000, and for the Earth's liquid metallic core it is higher still, about 10^6. For motions in the atmosphere of Jupiter, a planet with a diameter ten times that of the Earth and a rotation period as short as ten hours, the factor is more than 40. And even for the Sun, with a rotation period as long as a month but a diameter 10 times that of Jupiter, linear speeds of rotation greatly exceed typical relative speeds of large-scale atmospheric flow. So we must refer our dynamical equations to a more convenient reference frame.

If we choose a frame that rotates steadily with angular velocity $\boldsymbol{\Omega}$ relative to an inertial frame, (2.1) has to be modified by adding the so-called Coriolis force $2\rho\boldsymbol{\Omega}\times\boldsymbol{u}$ to the left-hand side and replacing $-\nabla V$ by $\boldsymbol{g}$, which now includes centripetal effects. When $\boldsymbol{\Omega}$ is so large in magnitude that $|\,2\rho\boldsymbol{\Omega}\times\boldsymbol{u}\,|$ greatly exceeds the terms $|\,\rho\partial\boldsymbol{u}/\partial t\,|$ and $|\,\rho(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\,|$ on typical time and space scales of the whole system, to a first approximation we can write

$$2\rho\boldsymbol{\Omega}\times\boldsymbol{u}\approx\boldsymbol{\Gamma}+\boldsymbol{F}, \tag{3.1}$$

where

$$\boldsymbol{\Gamma}\equiv-\nabla p+\boldsymbol{g}\rho. \tag{3.2}$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

When $| \boldsymbol{F} | \ll | 2\rho\boldsymbol{\Omega} \times \boldsymbol{u} |$, (3.1) reduces to

$$2\rho\boldsymbol{\Omega} \times \boldsymbol{u} \approx \boldsymbol{\Gamma}, \tag{3.3}$$

which expresses approximate 'geostrophic' balance between Coriolis forces and dynamic pressure gradients in the horizontal. To a good approximation, large-scale flows occupying extensive regions in the oceans and in the atmospheres of the Earth and other planets (with the exception of slowly rotating Venus) are either found or expected to be in geostrophic balance nearly everywhere. Exceptional regions are near the equator, where the vertical component of $\boldsymbol{\Omega}$ vanishes, and within narrow 'fronts' and jet streams, where the nonlinear advective term, $\rho(\boldsymbol{u} \cdot \nabla)\boldsymbol{u}$, is typically comparable with $2\rho\boldsymbol{\Omega} \times \boldsymbol{u}$ in magnitude. Indeed, the existence of such 'detached shear layers' in rapidly rotating fluids is implied by (3.3), which, being of lower order than the full equation, is diagnostic in character rather than prognostic, and therefore incapable of giving solutions that satisfy all the necessary boundary conditions (see Hide 1977). When the Lorentz force $\boldsymbol{j} \times \boldsymbol{B}$ makes the dominant contribution to $\boldsymbol{F}$ in (3.1) and is not negligible in magnitude in comparison with the Coriolis term, the equation becomes:

$$2\rho\boldsymbol{\Omega} \times \boldsymbol{u} \approx \boldsymbol{\Gamma} + \boldsymbol{j} \times \boldsymbol{B}. \tag{3.4}$$

This diagnostic equation expresses 'magnetostrophic' balance of forces acting on individual fluid elements. Such balance nearly everywhere may be the main property that characterizes large-scale MHD flows in the electrically conducting deep interiors of the planets, and also in the Sun and other stars.

Equation (3.3) is the basis of the meteorologist's 'Buys–Ballot law' – stand with your back to the wind in the Northern (Southern) Hemisphere and the low pressure is found to be on the left (right)! It follows from (3.3) that if the pattern of fluid flow $\boldsymbol{u}$ and the associated fields of p and ρ are symmetric about the axis of rotation, then the component of $\boldsymbol{u}$ perpendicular to the rotation axis is everywhere zero. This implies that any flow that is both axisymmetric and geostrophic cannot advect heat (or any other quantity) in directions that are perpendicular to the rotation axis. Such flows would therefore make no contribution to the overall heat balance of systems subject to axisymmetric applied heating and cooling. This result suggests, and experience based on many detailed studies of rapidly rotating laboratory systems and natural systems confirms (see Hide 1977), that large-scale non-axisymmetric flow capable of advecting heat perpendicularly to the rotation axis would develop in 'generic' (i.e. typical) cases. We thus see a connection between the basic function of thermally driven flows – namely heat transfer – and symmetry breaking which, from a theoretical point of view, can be associated with bifurcations produced by the action of Coriolis forces when $| \boldsymbol{\Omega} |$ is large enough.

This symmetry breaking associated with geostrophy and heat transfer in generic systems provides a good starting point in the discussion of the role of rotation in the production of magnetic fields by MHD dynamo action in astronomical bodies such as the Earth and other planets (see Hide 1982). By a theorem due to Cowling and others (for references see Proctor & Gilbert 1994) no magnetic field with an axis of symmetry can be maintained by fluid motions against Ohmic dissipation, so that a strictly axisymmetric pattern of fluid motion $\boldsymbol{u}$ and magnetic field $\boldsymbol{B}$ would be incapable of dynamo action. It is only possible to find non-decaying solutions to (2.5) (with $\boldsymbol{Z} = 0$) when the configuration of $\boldsymbol{B}$ has no axis of symmetry.

A magnetic field produced by self-exciting dynamo action can be regarded as having originated as an instability involving the amplification of a much weaker adventitious field through inductive action produced by fluid motion. Equilibration occurs when on average the rate at which buoyancy forces do work on the system is balanced by Ohmic and other types of energy dissipation. When the magnetic field is so weak that Lorentz forces are negligible, quasi-geostrophic non-axisymmetric flow is expected to occur. Dynamo action then amplifies the magnetic field, and various lines of evidence indicate that the field strength would typically build up until $\boldsymbol{j} \times \boldsymbol{B}$ is comparable in magnitude with $2\rho\boldsymbol{\Omega} \times \boldsymbol{u}$ (see (3.3) and (3.4)), a process by which geostrophic balance gives way to magnetostrophic balance.

We now turn to an important general property of geophysical fluids which bears directly on their response to any kind of forcing, namely their ability to support wave motions and oscillations in which material particle displacements possess components parallel to the wave fronts (see Acheson & Hide 1973; Gill 1984; Hide & Stewartson 1972; Lighthill 1978). Ordinarily, the essential mechanical difference between a fluid and a solid is the inability of the former to resist an applied shear stress, rendering it unable to transmit energy and information by shear waves. But 'elastoid' shear oscillations are possible in geophysical fluids owing to the action of Coriolis forces, Lorentz forces and, when the density distribution is 'bottom heavy', buoyancy (Archimedes!) forces, associated respectively with general rotation, the presence of magnetic fields, and gravity. Elastoid oscillations are often generated by internal instabilities, and their properties, which include anisotropy and dispersion, are influenced by mutual interactions and by interactions with background flows and bounding surfaces. Nonlinear effects do not always increase the degree of disorder in a system, for in the remarkable soliton phenomenon nonlinear advection exactly cancels effects due to linear wave dispersion.

The elastoid oscillations encountered in studies of the oceans and of the atmospheres of the Earth and other planets include the so-called Rossby waves and Kelvin waves. Complex interactions between waves involving Coriolis and buoyancy forces and generated by internal instabilities are also seen in the non-axisymmetric flow régimes investigated in laboratory work on sloping convection. Even richer in variety are the elastoid oscillations encountered in studies of the electrically conducting fluid interiors of planets and stars, where Lorentz forces arise. Of particular importance in magnetohydrodynamics is a special class of slow quasi-magnetostrophic oscillations in which Lorentz and Coriolis forces act in opposite directions. In the liquid metallic core of the Earth, the timescales of such oscillations could be comparable with those characteristic of the geomagnetic secular variation (decades to centuries), observations of which have been studied, often with a view to prediction, by geophysicists since the times of Halley and other early investigators (see § 1 above).

This concludes the general introduction to chaos in geophysical fluids. In Part II (Hide 1994), as mentioned in the summary, two representative areas of research are discussed, both concerned with phenomena attributable to symmetry-breaking bifurcations caused by gyroscopic (Coriolis) forces. These are (a) waves and eddies in the atmospheres of the Earth, Jupiter and Saturn (where, excep-

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

tionally laboratory experiments have been influential), and (*b*) MHD dynamos, where effective laboratory experiments are not technically feasible.

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